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Solutions to the Nonlinear Newell Equation

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Abstract

Many kinds of explicit and exact solutions of the nonlinear Newell equation, including the solitary wave solution, the singular traveling wave solution, and the triangle function-type periodic wave solutions, are presented by a direct trial function approach.

Subject Areas

Mathematical Analysis

Keywords

Nonlinear Newell Equation, Direct Trial Function Approach, Travelling Wave Solution, Solitary Wave Solution

1. Introduction

Exact solutions for nonlinear evolution equations (NEEs for short) play a very important part in nonlinear physical science since these solutions not only may well describe various natural phenomena, such as vibrations, solitons, and propagation with a finite speed, but also may give us insight into the physics aspects of the problems [1] [2]. For this reason, the construction of the explicit and exact solutions of NEEs has become one of the most important and essential tasks in nonlinear physics science. On account of the complexity of a nonlinear system, it is often difficult to seek the explicit and exact solutions of a real nonlinear physical model. Fortunately, many powerful methods for finding explicit and exact solutions to NEEs have been put forward. Among them are the hyperbolic tangent function expansion method [3] [4], the trial function method [5] [6], the auxiliary equation method [7] [8], the combination method [9], the auxiliary equation method [10], and the like. However, not all the above methods are universally suitable for solving all kinds of NEEs. As a result, it is still a very significant task to seek various powerful and efficient methods to solve NEEs.

In the present paper, our interest is in the investigation of the explicit and exact traveling wave solutions for the nonlinear Newell equation by utilizing a direct trial function method. As a consequence, a series of explicit and exact solutions to the nonlinear Newell equation are easily obtained.

2. Solutions for the Nonlinear Newell Equation

The celebrated nonlinear Newell equation under consideration reads

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\beta\frac{\partial^3 u}{\partial x^3}-\delta\frac{\partial^5 u}{\partial x^5}=0 \tag{1}$$

which arises in many different fields, such as shallow water models, plasma science, biophysics, and so on.

Recently, Xie and Tang [6] have proposed a unified trial function method to search for the explicit and exact solutions of three NEEs. This method seems to be slightly complicated in that it contains two trial functions. In order to improve this method and use it to solve Equation (1) easily, here we straightforwardly assume the ansatz solution in the following form

$$u=\frac{ae^{k_1\xi}}{\left(b+e^{k\xi}\right)^d},\quad 0\le k_1\le kd \tag{2}$$

where $\xi=x-ct$, and a,b,c,d,k,k_1 are undetermined constants.

To begin with, let us determine the constant d by means of partial balance between the highest order derivative terms and the highest degree nonlinear terms. Thanks to $0\le k_1\le kd$, we can take the order of u as

$$O(u)=d \tag{3}$$

then it is not hard to derive that

$$O\left(\frac{\partial^n u}{\partial x^n}\right)=d+n \tag{4}$$

The partial balance between the highest order derivative terms and the highest degree nonlinear terms in Equation (1) results in

$$O\left(\frac{\partial^4 u}{\partial x^4}\right)=O\left(u\frac{\partial u}{\partial x}\right) \tag{5}$$

from which it follows that

$$d=4 \tag{6}$$

So the ansatz solution (2) can be rewritten as

$$u=\frac{ae^{k_1\xi}}{\left(b+e^{k\xi}\right)^4},\quad 0\le k_1\le 4k \tag{7}$$

In view of Equation (7), it is readily deduced that

$$\frac{\partial u}{\partial t}=\frac{ace^{k_1\xi}\left[\left(4k-k_1\right)e^{k\xi}-bk_1\right]}{\left(b+e^{k\xi}\right)^5} \tag{8}$$

$$\frac{\partial u}{\partial x}=-\frac{a\mathrm{e}^{k_1\xi}\left[\left(4k-k_1\right)\mathrm{e}^{k\xi}-bk_1\right]}{\left(b+\mathrm{e}^{k\xi}\right)^5} \tag{9}$$

$$\begin{aligned}\frac{\partial^3 u}{\partial x^3}=&\frac{a\mathrm{e}^{k_1\xi}\left[b^3k_1^3+\left(3b^2k_1^3-4b^2k^3-12b^2k^2k_1-12b^2kk_1^2\right)\mathrm{e}^{k\xi}\right]}{\left(b+\mathrm{e}^{k\xi}\right)^7}\\&+\frac{a\mathrm{e}^{k_1\xi}\left[\left(52bk^3+36bk^2k_1-24bkk_1^2+3bk_1^3\right)\mathrm{e}^{2k\xi}+\left(48k^2k_1-64k^3-12kk_1^2+k_1^3\right)\mathrm{e}^{3k\xi}\right]}{\left(b+\mathrm{e}^{k\xi}\right)^7}\end{aligned} \tag{10}$$

$$\begin{aligned}\frac{\partial^5 u}{\partial x^5}=&\frac{a\mathrm{e}^{k_1\xi}\left[b^5k_1^5-\left(4b^4k^5+20b^4k^4k_1-40b^4k^3k_1^2-40b^4k^2k_1^3-20b^4kk_1^4-5b^4k_1^5\right)\mathrm{e}^{k\xi}\right]}{\left(b+\mathrm{e}^{k\xi}\right)^9}\\&+\frac{a\mathrm{e}^{k_1\xi}\left(284b^3k^5+620b^3k^4k_1+40b^3k^2k_1^3+440b^3k^3k_1^2-80b^3kk_1^4+10b^3k_1^5\right)\mathrm{e}^{2k\xi}}{\left(b+\mathrm{e}^{k\xi}\right)^9}\\&+\frac{a\mathrm{e}^{k_1\xi}\left(-1620b^2k^4k_1-2124b^2k^5+360b^2k^3k_1^2+360b^2k^2k_1^3-120b^2kk_1^4+10b^2k_1^5\right)\mathrm{e}^{3k\xi}}{\left(b+\mathrm{e}^{k\xi}\right)^9}\\&+\frac{a\mathrm{e}^{k_1\xi}\left(3284bk^5-980bk^4k_1-760bk^3k_1^2+440bk^2k_1^3-80bkk_1^4+5bk_1^5\right)\mathrm{e}^{4k\xi}}{\left(b+\mathrm{e}^{k\xi}\right)^9}\\&+\frac{a\mathrm{e}^{k_1\xi}\left(-1024k^5+1280k^4k_1-640k^3k_1^2+160k^2k_1^3-20kk_1^4+k_1^5\right)\mathrm{e}^{5k\xi}}{\left(b+\mathrm{e}^{k\xi}\right)^9}\end{aligned} \tag{11}$$

Substituting Equations (7)-(11) into Equation (1) yields

$$\begin{aligned}&ab^5k_1\left(k_1^2\beta-k_1^4\delta-c\right)\mathrm{e}^{k_1\xi}+a^2bk_1\mathrm{e}^{2k_1\xi}+ab^4\left(4ck-5ck_1-4k^3\beta-12k^2k_1\beta\right.\\&\left.-12kk_1^2\beta+5k_1^3\beta+4k^5\delta+20k^4k_1\delta+40k^3k_1^2\delta+40k^2k_1^3\delta+20kk_1^4\delta-5k_1^5\delta\right)\mathrm{e}^{(k+k_1)\xi}\\&+2ab^3\left(8ck-5ck_1+22k^3\beta+6k^2k_1\beta-24kk_1^2\beta+5k_1^3\beta-142k^5\delta-310k^4k_1\delta\right.\\&\left.-220k^3k_1^2\delta-20k^2k_1^3\delta+40kk_1^4\delta-5k_1^5\delta\right)\mathrm{e}^{(2k+k_1)\xi}+2ab^2\left(12ck-5ck_1+18k^3\beta\right.\\&+54k^2k_1\beta-36kk_1^2\beta+5k_1^3\beta+1062k^5\delta+810k^4k_1\delta-180k^3k_1^2\delta-180k^2k_1^3\delta\\&\left.+60kk_1^4\delta-5k_1^5\delta\right)\mathrm{e}^{(3k+k_1)\xi}+ab\left(16ck-5ck_1-76k^3\beta+132k^2k_1\beta-48kk_1^2\beta\right.\\&\left.+5k_1^3\beta-3284k^5\delta+980k^4k_1\delta+760k^3k_1^2\delta-440k^2k_1^3\delta+80kk_1^4\delta-5k_1^5\delta\right)\mathrm{e}^{(4k+k_1)\xi}\\&+a\left(4ck-ck_1-64k^3\beta+48k^2k_1\beta-12kk_1^2\beta+k_1^3\beta+1024k^5\delta-1280k^4k_1\delta\right.\\&\left.+640k^3k_1^2\delta-160k^2k_1^3\delta+20kk_1^4\delta-k_1^5\delta\right)\mathrm{e}^{(5k+k_1)\xi}+a^2\left(k_1-4k\right)\mathrm{e}^{(k+2k_1)\xi}=0\end{aligned} \tag{12}$$

We find that only when $k_1=2k$, Equation (1) has a nontrivial solution. Under this condition, Equation (12) can be rewritten as

$$\begin{aligned}&2ab^5k\left(4k^2\beta-c-16k^4\delta\right)\mathrm{e}^{2k\xi}-6ab^4k\left(c+6k^2\beta-114k^4\delta\right)\mathrm{e}^{3k\xi}\\&+2abk\left(a-2b^2c-22b^2k^2\beta-1322b^2k^4\delta\right)\mathrm{e}^{4k\xi}\\&-2ak\left(a-2b^2c-22b^2k^2\beta-1322b^2k^4\delta\right)\mathrm{e}^{5k\xi}\\&+6abk\left(c+6k^2\beta-114k^4\delta\right)\mathrm{e}^{6k\xi}-2ak\left(4k^2\beta-c-16k^4\delta\right)\mathrm{e}^{7k\xi}=0\end{aligned} \tag{13}$$

Because of the arbitrariness of ξ, Equation (13) engenders the following algebraic equations

$$4k^2\beta - c - 16k^4\delta = 0 \tag{14}$$

$$c + 6k^2\beta - 114k^4\delta = 0 \tag{15}$$

$$a - 2b^2c - 22b^2k^2\beta - 1322b^2k^4\delta = 0 \tag{16}$$

Solving the above system of equations, we find that

$$a = \frac{1680b^2\beta^2}{169\delta},\ c = \frac{36\beta^2}{169\delta},\ k = \pm\frac{\sqrt{\beta}}{\sqrt{13\delta}},\ b = \text{arbitrary constant} \tag{17}$$

Plugging Equation (17) into Equation (7), we obtain the general traveling wave solution to the nonlinear Newell Equation (1) as follows

$$u = \frac{16805b^2\beta^2\mathrm{e}^{2k\xi}}{169\delta\left(b+\mathrm{e}^{2k\xi}\right)^4} + c - \frac{36\beta^2}{169\delta} \tag{18}$$

Making use of the following identity

$$\frac{\mathrm{e}^{2x}}{\left(\mathrm{e}^{2x}+1\right)^2} = \frac{1}{4}\operatorname{sech}^2 x \tag{19}$$

and setting $b=1$ in Equation (18), we obtain the solitary wave solution to the nonlinear Newell Equation (1) as follows

$$u = \frac{105\beta^2}{169\delta}\operatorname{sech}^4 k\xi + c - \frac{36\beta^2}{169\delta} \tag{20}$$

Which may help explain nonlinear wave phenomena of diffusion in fluid mechanics.

Making use of the following identity

$$\operatorname{sech}^2 x = \frac{2}{\cosh 2x + 1} \tag{21}$$

then Equation (20) can be converted to

$$u = \frac{420\beta^2}{169\delta}\left(\frac{1}{\cosh 2k\xi + 1}\right)^2 + c - \frac{36\beta^2}{169\delta} \tag{22}$$

Similarly, making use of the following identity

$$\frac{\mathrm{e}^{2x}}{\left(\mathrm{e}^{2x}-1\right)^2} = \frac{1}{4}\operatorname{csch}^2 x \tag{23}$$

and setting $b=-1$ in Equation (18), we get the singular traveling wave solution to the nonlinear Newell Equation (1) as follows

$$u = \frac{105\beta^2}{169\delta}\operatorname{csch}^4 k\xi + c - \frac{36\beta^2}{169\delta} \tag{24}$$

Making use of the following identity

$$\operatorname{csch}^2 x = \frac{2}{\cosh 2x - 1} \tag{25}$$

then Equation (24) can be rewritten as

$$u = \frac{420\beta^2}{169\delta}\left(\frac{1}{\cosh 2k\xi - 1}\right)^2 + c - \frac{36\beta^2}{169\delta} \tag{26}$$

Let

$$k = ik' \tag{27}$$

where i is the imaginary unit, and k' the constant and making use of the following two identities

$$\operatorname{sech}(ix) = \sec x, \quad \operatorname{csch}(ix) = -i\csc x \tag{28}$$

then Equation (22) and Equation (24) can be reduced to

$$u = \frac{105\beta^2}{169\delta}\sec^4 k'\xi + c - \frac{36\beta^2}{169\delta} \tag{29}$$

$$u = \frac{105\beta^2}{169\delta}\csc^4 k'\xi + c - \frac{36\beta^2}{169\delta} \tag{30}$$

which are two triangle function-type periodic wave solutions of the nonlinear Newell Equation (1).

Finally, it is worthwhile to point out that if each "*c*" in the above solutions is replaced by "-*c*", then we can obtain other lots of explicit and exact traveling wave solutions to the nonlinear Newell equation. Here, we do not list them one by one for the limit of length.

3. Conclusions and Remarks

To sum up, by utilizing a direct trial function method, many types of explicit and exact solutions for the nonlinear Newell equation are obtained. Thanks to the nonlinear Newell equation being a real physical model, solving it has been paid much attention by many authors, and quite a few papers have investigated its traveling wave solutions. However, compared with some proposed approaches in the literature, the above technique described herein appears to be relatively concise, straightforward, and less calculative. For example, the method proposed in Ref. [6] contains two trial functions, and it needs to perform more tedious calculations. However, my approach has only one trial function, and it certainly needs to perform fewer calculations. In addition, it should be pointed out that the solutions found herein definitely have physical meaning. For example, the solitary wave solution (20) for the nonlinear Newell Equation (1) may help to explain the nonlinear wave phenomena of diffusion in fluid mechanics.

Conflicts of Interest

The author declares no conflicts of interest.

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